Control of Time-Periodic Systems

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The problem of active control of linear, time-periodic systems is discussed and several useful pole placement techniques are found. The techniques available for solving a linear, time-periodic system (a Floquet problem) are summarized. The problem of determining the state feedback necessary to alter one characteristic root is solved completely. When conjugate roots are to be moved, a useful relation is found linking the old and new roots in the case of scalar control, and in the case of vector control, we present a pole-placement algorithm. Finally, we offer some discussion on the remaining problems of time-periodic control.

Introduction

THE study of time-periodic linear systems dates to the last decades of the last century, when Floquet discovered the fundamental form of the solution for such problems. The first, and still most extensive, application of his results was in the field of celestial mechanics. The literature of this field is far too extensive to quote here; we mention only the seminal work of Poincaré[1] and Szebehely[2] for a review of applications to the restricted problem of three bodies. The usual use for Floquet theory is in deciding the stability of a known periodic orbit. Other applications are less common. In particular, control of unstable periodic orbits has been discussed, to our knowledge, only by Breakwell et al.[3] and by Wiesel and Shelton.[4] The use of Floquet theory as a starting point for perturbation theory has been treated by Wiesel.[5] However, the main use of Floquet theory has been the treatment of the stability problem alone.

Time-periodic systems arise in other fields as well. The stability of a spinning unsymmetrical satellite in a circular orbit was treated by Kane and Shippey,[6] and the symmetrical satellite in an elliptical orbit was discussed by Kane and Barba.[7] Both of these works were concerned only with mapping stability regions, and no control was considered. The inclusion of control for these problems was treated by Calico and Yeakel.[8] Time-periodic stability issues also arise in the problem of the helicopter rotor, where stability issues have been treated by Peters and Hohenemser,[9] and Hohenemser and Yin.[10] The use of Floquet theory to actually design the helicopter control system has yet to be performed. Note, however, that any linearized rotating machinery problem (where a simple rotating reference frame cannot remove the time dependence) can be treated as a Floquet problem.

There is a very extensive literature on control theory, of course, but most frequently it is the case of linear, constant-coefficient systems which is treated. In most instances, the constant-coefficient methods do not generalize in a simple way to the case of time-periodic coefficients. In this paper we will summarize the results of several investigations which we have undertaken in the last few years, and try to present a coherent picture of the status of the pole placement problem for time-periodic systems. We first bring together the complete algorithm for actually solving Floquet systems and then consider the case of simple scalar control. We then pass to the case of vector control, and offer a new algorithm for pole placement in this case. Finally, we offer a discussion of what yet remains to be done on this problem.

Received Feb. 28, 1983; revision received July 11, 1983. This paper is declared a work of the U.S. Government and therefore is in the public domain.

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Construction of a Floquet Solution

Constructing the numerical solution to a set of linear, time-periodic differential equations

$$\frac{d}{dt}x = A(t)x \tag{1}$$

where $A(t)$ is periodic with period T, is only slightly more difficult than solving the constant-coefficient case. The procedures differ, however, so a complete discussion is included here. Since Eq. (1) is linear, its solution is given by the state transition matrix $\Phi(t,0)$, as

$$x(t) = \Phi(t,0)x(0) \tag{2}$$

where $\Phi(t,0)$ satisfies the usual differential equation and initial conditions

$$\frac{d}{dt}\Phi(t,0) = A(t)\Phi(t,0) \tag{3}$$

$$\Phi(0,0) = I$$

The main result of Floquet is that $\Phi(t,0)$ can be factored into two matrices, F and J, in the form

$$\Phi(t,0) = F(t)e^{Jt}F^{-1}(0) \tag{4}$$

The matrix J is a constant matrix, and is usually most conveniently put in the Jordan normal form. Its diagonal entries ω_i are termed Poincaré exponents, the analog of eigenvalues for constant-coefficient systems. The matrix $F(t)$ is periodic with the same period T as the original system, Eq. (1). In a constant-coefficient system the matrix F would be the constant matrix of system eigenvectors. In fact, the *only* difference between the solution, Eq. (4), to a Floquet problem and the solution to a constant-coefficient problem is that the eigenvector matrix F becomes time-periodic in the former case. Solving a Floquet problem for all time thus requires finding the constant matrix J and the periodic matrix $F(t)$ over one period.

Since the matrix F is periodic, $F(0)=F(T)$, and at $t=T$, Eq. (4) becomes

$$\Phi(T,0) = F(0)e^{JT}F^{-1}(0) \tag{5}$$

Thus, $F(0)$ is the matrix of eigenvectors of the monodromy matrix $\Phi(T,0)$. The only general technique for constructing the monodromy matrix appears to be direct numerical integration of Eq. (3) for one period. Also, the eigenvalues of the monodromy matrix λ_i are related to the Poincaré exponents ω_i by

$$\lambda_i = \exp(\omega_i T) \tag{6}$$

a result that is confirmed by inspection of Eq. (5). So, at the cost of numerically integrating Eq. (3) for one period, the system Poincaré exponents and the eigenvector matrix at $t = 0$ may be obtained. The eigenvectors $F(t)$ may be normalized at $t = 0$, but since they are periodic functions of time, they will not stay normalized over the entire period.

It is very common in the literature to stop at this point. Since $F(t)$ is periodic, and therefore bounded, the stability of the system is governed by the ω_i (or λ_i) alone, and this information is now available. However, construction of the complete solution, Eq. (4), requires knowledge of $F(t)$ over one period. Substituting Eq. (4) into Eq. (3) and rearranging results in

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t) = A(t)F(t) - F(t)J \tag{7}$$

Any two matrices $A(t)$ and J related as in Eq. (7) are said to be kinematically similar. In the present case, since $A(t)$ is periodic, the matrix J is a constant matrix. Hence, any periodic matrix is kinematically similar to a constant matrix. It is of interest to note that if $A(t)$ is a constant matrix, Eq. (7) defines a constant F, and the matrices A and J are similar in the conventional sense. Initial conditions for Eq. (7) are already available from the eigenvectors found in Eq. (5). The results of numerically integrating Eq. (7) over one period can be reduced to a convenient, usable form by harmonic analysis. A short summary of this invaluable technique is included in Ref. 11. With the construction of J and $F(t)$, the complete solution to the system, Eq. (1), is available for all time.

However, the above results may be inconvenient, since both J and $F(t)$ may be complex matrices. Appropriate rearrangements of J and F can be found which make both matrices real. In particular, F should be constructed of column vectors f_i, which are either 1) the real-valued eigenvector associated with a real ω_i, or 2) the pair of vectors $f_{i\,\mathrm{real}}$, $f_{i\,\mathrm{imag}}$, the real and imaginary parts, respectively, of an eigenvector associated with a complex pair of ω_i. The matrix J will no longer be in Jordan normal form, but will consist of either 1) diagonal real entries ω_i, or 2) diagonal blocks of the form:

$$\begin{bmatrix} \mathrm{Re}(\omega_i) & \mathrm{Im}(\omega_i) \\ -\mathrm{Im}(\omega_i) & \mathrm{Re}(\omega_i) \end{bmatrix} \tag{8}$$

for a pair of complex ω_i. In the case of complex ω_i, the real parts are placed on the diagonal of a block such as Eq. (8), with the imaginary parts off the diagonal as shown in Eq. (8). The matrix $\exp(Jt)$ will then contain diagonal entries $\exp(\omega_i t)$ for real entries, or blocks of the form:

$$\exp(\mathrm{Re}(\omega_i)t)\begin{bmatrix} \cos\mathrm{Im}(\omega_i)t & -\sin\mathrm{Im}(\omega_i)t \\ \sin\mathrm{Im}(\omega_i)t & \cos\mathrm{Im}(\omega_i)t \end{bmatrix} \tag{9}$$

for blocks of J of the form of Eq. (8). With these rearrangements, all previous formulae are completely unaltered.

It is often necessary to use the inverse matrix $F^{-1}(t)$ in the developments to follow. Since the state transition matrix is never singular, $F(t)$ is always invertible. One can obtain $F^{-1}(t)$ by inverting F at a given value of t, but this method is expensive and can greatly increase roundoff error. Alternately, if the identity $FF^{-1} = I$ is differentiated, and substituted from Eq. (7), the result is

$$\frac{\mathrm{d}}{\mathrm{d}t}(F^{-1}) = -F^{-1}(t)A(t) + JF^{-1}(t) \tag{10}$$

So, $F^{-1}(t)$ can be constructed by numerical integration and reduced to its Fourier series coefficients by the same techniques used for $F(t)$ itself.

The inverse of $F(t)$ can also be related to the time-dependent modal matrix associated with the system of Eq. (1) Consider the adjoint problem

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = -A(t)^T y(t) \tag{11}$$

Denoting the modal matrix associated with $-A^T$ as $L(t)$, one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t) = -A(t)^T L(t) + L(t)J \tag{12}$$

or, taking the transpose

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t)^T = -L(t)^T A(t) + JL(t)^T \tag{13}$$

By direct comparison of Eq. (13) with Eq. (10), one finds

$$F^{-1}(t) = L(t)^T \tag{14}$$

Therefore, the time-dependent eigenvectors of the adjoint problem are orthogonal to the time-dependent eigenvectors of the original system. It is also obvious that for a self-adjoint system the matrix $F(t)$ is orthonormal. The use of Eq. (13) is recommended as the preferred technique for the construction of $F^{-1}(t)$ from a numerical point of view.

Now, if we introduce new variables η, termed modal variables, by

$$x(t) = F(t)\eta(t) \tag{15}$$

then the time-periodic system of Eq. (1) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta = F^{-1}(t)\left[A(t)F(t) - \frac{\mathrm{d}}{\mathrm{d}t}F(t)\right]\eta \tag{16}$$

In view of Eq. (7), this reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta = J\eta \tag{17}$$

Thus, the eigenvector matrix $F(t)$ is a periodic transformation which reduces the time-periodic system of Eq. (1) to a constant-coefficient system, Eq. (17). We shall refer to the variables η as the modal variables for the system, since the transformation of Eq. (15) performs the same decoupling function as in the constant-coefficient case.

Modal Control Theory

The open-loop characteristics of Eq. (1) are determined by the Poincaré exponents for the system. These exponents can be changed by adding state variable feedback. To this end, consider the standard control problem

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = A(t)x(t) + B(t)u(t) \tag{18}$$

where the matrix $A(t)$ is that of Eq. (1) and the matrix $B(t)$ determines the distribution of control in the system. The control matrix B is assumed to be either constant or time-periodic with the same period T as the fundamental dynamical system. Assuming full state feedback, the control vector $u(t)$ is given by

$$u(t) = G(t)x(t) \tag{19}$$

where $G(t)$ is the gain matrix. Therefore, Eq. (18) becomes

$$\frac{d}{dt}x_c(t) = [A(t) + B(t)G(t)]x_c \qquad (20)$$

where x_c is the closed-loop state.

In terms of the modal variables η, Eq. (20) becomes

$$\frac{d}{dt}\eta_c = \left[J + F(t)^{-1}B(t)G(t)F(t)\right]\eta_c \qquad (21)$$

which is itself a Floquet problem if one assumes that $G(t)$ is, at worst, periodic with the system period T. The task remaining is to determine $G(t)$ such that the closed-loop system has acceptable properties. For the present analysis, a modal control technique will be developed for the case of periodic systems. Both scalar and vector control cases will be considered. We shall first consider the case of a positive real Poincaré exponent, and we will then turn our attention to complex conjugate roots.

Scalar Control—One Mode

For the case of a scalar control, Eq. (21) in modal variables is

$$\frac{d}{dt}\eta = J\eta + g(t)u(t) \qquad (22)$$

where $g(t)$ is the $n\times 1$ periodic mode controllability matrix defined by

$$g(t) = F(t)^{-1}B(t) \qquad (23)$$

Modes η_{ci} are controllable if the corresponding $g_i(t)$ is nonzero. Consider first the case where one Poincaré exponent ω_i is a positive real number, and the original system, Eq. (1), is thus unstable. In order to change just one Poincaré exponent, control of the form

$$u(t) = k(t)\eta_{ci} \qquad (24)$$

is chosen, where η_{ci} is the unstable modal variable. The controlled system equations take the form

$$\frac{d}{dt}\eta_c = \begin{bmatrix} \omega_1 & 0 & kg_1 & 0 \\ 0 & \omega_2 & kg_2 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & kg_i+\omega_i & 0 \\ 0 & 0 & kg_n & \omega_n \end{bmatrix}\eta_c \qquad (25)$$

From Eq. (25), it is obvious that (except for the ith element) the closed-loop system has the same Poincaré exponents as the open-loop system. It remains to be shown that $k(t)$ can be chosen such that ω_{ci} can be any desired value. To this end, consider the equation for η_{ci} alone

$$\frac{d}{dt}\eta_{ci} = [\omega_i + k(t)g_i(t)]\eta_{ci} \qquad (26)$$

Equation (26) can be solved by the use of an integrating factor to yield

$$\eta_{ci}(t) = \eta_i(0)\exp\left[\int_0^t(\omega_i + k(t)g_i(t))\,dt\right] \qquad (27)$$

The function $g_i(t)$ is periodic and may be expanded in a Fourier series. Separating this series into its constant term g_{ic} and its purely periodic part $g_{ip}(t)$, we consider constant gain k and substitute into Eq. (27) to obtain

$$\eta_{ci}(t) = \eta_i(0)\exp[(\omega_i + kg_{ic})t]\exp\left[\int_0^t(kg_{ip}(t))\,dt\right] \qquad (28)$$

The last exponential function is a purely periodic function of time, so the new Poincaré exponent is the argument of the first exponential, or

$$\omega_i' = \omega_i + kg_{ic} \qquad (29)$$

This simply indicates that the root shifts along the real axis as a linear function of the gain k.

This method is not restricted to the case where $g_i(t)$ has a constant part in its Fourier series. If one considers the case where

$$k(t) = k\sin\left[\frac{2n\pi t}{T}\right] \qquad (30)$$

the product $k(t)g_i(t)$ will generate a constant term via standard trigonometric identities. A cosine function may also be used in Eq. (30), of course, with the same result. In either case, the new Poincaré exponent will be given by

$$\omega_i' = \omega_i + \frac{kc_{ni}}{2} \qquad (31)$$

where c_{ni} is the coefficient of the nth sine or cosine term in the Fourier series expansion of $g_i(t)$. Again, the root shifts linearly with the gain k. Equation (30) is also easily extended to the case where the gain function is permitted to be a general Fourier series.

Equations (29) and (31) for the change in the new Poincaré exponent were developed by considering the state equations in terms of the modal coordinates. In terms of the physical coordinates, the control necessary to change just the ith mode as shown in Eq. (24) is given by

$$u(t) = k(t)f_i^{-1}(t)x(t) \qquad (32)$$

Here, f_i^{-1} is the ith row of the F^{-1} matrix. Furthermore, the time-dependent gain $k(t)$ is determined based on the Fourier series for $g_i(t)$. The entire operation, therefore, requires determining the ith time-dependent eigenvector of the adjoint problem, forming $g_i(t)$ and its associated Fourier series, and then choosing $k(t)$ as either a constant or in the form of Eq. (30). The control is then obtained from Eq. (32), and may be inserted into the original system, Eq. (1), and the Floquet solution repeated to verify the root shift.

The procedure outlined above allows for the movement of a single real exponent. This is, of course, an absolutely necessary process, since it allows for the removal of instabilities. However, the question remains as to how more than one exponent may be moved predictably. One way in which this can be done is by repeating the entire procedure on the new controlled system, which is itself a Floquet system. By a second modal transformation, the system of Eq. (1), augmented by the first control term, would be rediagonalized. A new set of eigenvectors would be determined by numerical integration. This is unattractive from the computational viewpoint, especially if several exponents must be moved. This approach is also not applicable to moving pairs of complex conjugate roots.

Scalar Control—Two Modes

We now turn to the case of controlling two modes at once. This could include the case of two real Poincaré exponents, or a complex conjugate pair of exponents. In order to understand this problem more clearly, consider the control given by

$$u = k(t)\eta \qquad (33)$$

where $k = [0,0,\ldots,k_i(t),\ldots,k_j(t),\ldots]$. By a simple reordering of the equations, the i and j elements may be made to occur consecutively. The closed-loop equations in modal form then become (with j replaced by $i+1$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta_c = \begin{bmatrix} \omega_1 & k_i g_1 & k_{i+1} g_1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \omega_i + k_i g_i & k_{i+1} g_i & 0 \\ 0 & k_i g_{i+1} & \omega_{i+1} + k_{i+1} g_{i+1} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & k_i g_N & k_{i+1} g_N & \omega_N \end{bmatrix} \eta_c \tag{34}$$

By inspection, the Poincaré exponents of Eq. (34) are those of the open-loop system, except for ω_i and ω_{i+1}. These two exponents are determined from the coupled equations for the two modes η_i and η_{i+1}. These two equations are decoupled from the remainder of Eq. (34), and may be separated to give the two-dimensional system

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta_c = \begin{bmatrix} \omega_1 + k_1(t) g_1(t) & k_2(t) g_1(t) \\ k_1(t) g_2(t) & \omega_2 + k_2(t) g_2(t) \end{bmatrix} \eta_c \tag{35}$$

The task at hand is to determine the gains $k_1(t)$ and $k_2(t)$ such that the closed-loop Poincaré exponents ω_1' and ω_2' have desired values. Recalling that the gains g_1 and g_2 can be expressed as Fourier series, and assuming that k_1 and k_2 are either constant, or are of the form in Eq. (30), Eq. (35) may be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta_c = A_1 \eta_c + A_2(t)\eta_c \tag{36}$$

where A_1 is a constant matrix, and $A_2(t)$ is a purely periodic matrix with no constant part. The gains may be easily chosen such that A_1 has any eigenvalues desired. Unfortunately, choosing the gains such that A_1 has negative eigenvalues does not assure stability of the system of Eq. (36). Furthermore, no general technique exists for finding the eigenvalues of Eq. (36) other than the general numerical method for solving a Floquet system, described in the first section of this paper. The technique used for moving one exponent, therefore, does not easily generalize to the case of moving more than one exponent.

Some insight into the choice of gains for this problem can be obtained by considering the well known relationship

$$\frac{\mathrm{d}}{\mathrm{d}t} D(t) = \mathrm{tr}(A(t)) D(t) \tag{37}$$

where $D(t) = \det[\Phi(t,0)]$ and $\mathrm{tr}(\cdot)$ is the trace. This has the solution

$$D(t) = D(0)\exp\left[\int_0^t \mathrm{tr}(A(t))\,\mathrm{d}t\right] \tag{38}$$

By evaluating at $t = T$, and noting that $\Phi(T,0)$ is the monodromy matrix, one finds

$$\prod_{i=1}^{N} \lambda_i = \exp\left[\int_0^T \mathrm{tr}(A(t))\,\mathrm{d}t\right] \tag{39}$$

in terms of the characteristic multipliers λ_i, or, in terms of the Poincaré exponents

$$\sum_{i=1}^{N} \omega_i = \frac{1}{T}\int_0^T \mathrm{tr}(A(t))\,\mathrm{d}t \tag{40}$$

For the case given in Eq. (35) we find

$$\omega_1' + \omega_2' = \omega_1 + \omega_2 + \frac{1}{T}\int_0^T [k_1(t) g_1(t) + k_2(t) g_2(t)]\,\mathrm{d}t \tag{41}$$

Assuming that g_1 and g_2 have constant terms in their Fourier series, and that constant gains are used, Eq. (41) becomes

$$\omega_1' + \omega_2' = \omega_1 + \omega_2 + k_1 g_{1c} + k_2 g_{2c} \tag{42}$$

This relationship may be used to set the sum of the new Poincaré exponents to any desired value. For stability it is necessary that the sum be nonpositive. This is, of course, not a sufficient condition, since it does not require each individual Poincaré exponent to have a negative real part. Equation (42) can be used to define pairs of k_1 and k_2 which yield a specific sum, and from these pairs the specific gains which yield the desired ω_1' and ω_2' may be determined by a systematic numerical search procedure. It is important to remember, however, that the monodromy matrix for Eq. (35) will have to be determined numerically for each pair of gains in order to determine the actual Poincaré exponents. In the case where the original Poincaré exponents are complex conjugate, Eq. (42) determines only the sum of their real parts. The actual new exponents may still have imaginary parts, or they may both be real.

Vector Control—Two Modes

We now turn to the case where we have at our disposal more than one dimension for the control vector $u(t)$. For simplicity, we restrict ourselves to two dimensions, both for the control vector and for the modal state vector η. This is not a loss of generality, since the extra dimensions in a larger order gain matrix may obviously be used to decouple the control from all but the two modes in question. The control term in Eq. (33) now takes the form

$$u(t) = k(t)\eta_c = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \eta_c \tag{43}$$

where the four gain elements k_{ij} are all functions of time. Inserting this control term into the two-dimensional system, Eq. (35), now written as a pair of complex-conjugate roots, one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta_c = \begin{bmatrix} k_{11} g_{11} + k_{21} g_{12} + \mathrm{Re}(\omega_i) & k_{12} g_{11} + k_{22} g_{12} + \mathrm{Im}(\omega_i) \\ k_{11} g_{21} + k_{21} g_{22} - \mathrm{Im}(\omega_i) & k_{12} g_{21} + k_{22} g_{22} + \mathrm{Re}(\omega_i) \end{bmatrix} \eta_c \tag{44}$$

For the case of the present example, we shall transform these two oscillatory modes into a pair of uncoupled, purely damped modes. We dispose of two of the four gain functions k_{ij} by explicitly decoupling the two modes in question. This gives us the two equations

$$\begin{aligned} k_{11}(t) g_{21}(t) + k_{21}(t) g_{22}(t) &= \mathrm{Im}(\omega_i) \\ k_{12}(t) g_{11}(t) + k_{22}(t) g_{12}(t) &= -\mathrm{Im}(\omega_i) \end{aligned} \tag{45}$$

We now pick two negative real numbers, $-\omega_1'$ and $-\omega_2'$, and force the diagonal terms above to assume these values. This gives us two more equations

$$\begin{aligned} k_{11}(t) g_{11}(t) + k_{21}(t) g_{12}(t) &= -\omega_1' - \mathrm{Re}(\omega_i) \\ k_{12}(t) g_{21}(t) + k_{22}(t) g_{22}(t) &= -\omega_2' - \mathrm{Re}(\omega_i) \end{aligned} \tag{46}$$

These relations constitute four linear equations in the four unknown functions k_{ij}. At any given time the numerical values of the known functions g_{ij} may be calculated, and the values of the k matrix may be obtained. Performing this calculation for many evenly spaced values of time over one period T yields sufficient data for the harmonic analysis algorithm, and the Fourier series representation of the k matrix may be obtained. In order for this process to be free from infinite values of k, it is necessary that the determinant of the coefficients of the four equations never vanish. That is, the controllability condition is given by

$$[g_{11}(t)g_{22}(t)-g_{12}(t)g_{21}(t)]^2\neq 0 \tag{47}$$

at any time during one period.

Although we have shown the original system as having a complex-conjugate pair of roots in this example, it is not difficult to construct different cases. Of equal interest would be the case of two positive real Poincaré exponents, which may be handled with equal ease by the above technique. The only change, in fact, would be in Eq. (45), whose right sides would be zero. The controllability condition would not change from the form given in Eq. (47). Note, however, that the technique outlined above is not the most general form of the control law for a two-mode time-periodic system. We have imposed the condition that the control decouple the two new modes in the modal variables of the old system. This was done, of course, to eliminate the necessity to numerically calculate the monodromy matrix of the system, Eq. (44), which process would be forced upon us if the modes are allowed to remain coupled. Also, this technique is somewhat inelegant, in that it forces us to use two control dimensions if we simply want to supply damping to a pair of conjugate imaginary Poincaré exponents.

Vector Control—General Case

We now turn to the case of a general dynamical system of order N. It is typical of such systems that only momentum states are directly accessible to any control device. This implies that the controllability matrix g will have maximum rank $N/2$. We partition the J matrix into the matrix of modes we wish to control, J_c, and the modes we will ignore, J_i, with a similar partition for the η vector. The gain matrix k is partitioned as

$$u(t)=k(t)\eta=[k_1(t)|k_2(t)]\eta \tag{48}$$

The modal controllability matrix $g(t)$ becomes

$$g(t)=F^{-1}(t)B(t)=\begin{bmatrix} g_1(t)\\ \cdots \\ g_2(t)\end{bmatrix} \tag{49}$$

where the corresponding rearrangements must be made in the matrix F^{-1}. All submatrices are now square, and of order $N/2$. The general control problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta=J\eta+gk\eta \tag{50}$$

then becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\eta_c\\ \eta_i\end{pmatrix}=\begin{bmatrix} J_c+g_1k_1 & g_1k_2\\ g_2k_1 & J_i+g_2k_2\end{bmatrix}\begin{pmatrix}\eta_c\\ \eta_i\end{pmatrix} \tag{51}$$

It is immediately apparent that the choice of

$$k_2(t)\equiv[0] \tag{52}$$

has the effect of decoupling the controlled modes from the modes we wish to ignore. It also leaves the Poincaré exponents of the ignored modes, J_i, unaltered, since the remaining coupling term above alters only the eigenvectors of the complete system.

We may now specify that the controlled modes are to have Poincaré exponents given by the matrix J_c'. Imposing this condition yields

$$J_c+g_1(t)k_1(t)=J_c' \tag{53}$$

This gives us the necessary control law

$$k_1(t)=[g_1(t)]^{-1}(J_c'-J_c) \tag{54}$$

as an explicitly periodic function of time. The controllability condition is that the inverse of the matrix $g_1(t)$ exist over the entire interval from $t=0$ to $t=T$. We note that this condition is exactly the equivalent of Eq. (47) in the two mode case. If this condition is met, the time-periodic gain matrix k_1 may be constructed by the same numerical techniques sketched in the last section. Since the matrix B is assumed to be of full column rank $N/2$, the question of meeting the controllability condition becomes dependent on the particular rearrangement of F^{-1} used, and, hence, is a function of the particular modes chosen for control.

The above results may be extended somewhat if we do not specify J_c' in the modified Jordan form, but instead use a diagonal matrix of Fourier series, or even an upper triangular matrix of Fourier series. This is sufficient to decouple the controlled modes into $N/2$ systems of the form of Eq. (26). The new Poincaré exponents would simply be the constant terms in the Fourier series placed in the diagonal slots. The additional freedom this permits could then be disposed of by imposing some form of optimization requirement on the control vector $u(t)$. This emphasizes again that our results are only a subset of possible gain selection techniques, and do not represent the most general possibilities.

Summary

The active control of linear, time-periodic systems has been considered, and several pole placement techniques have been developed. Specifically, using scalar control, a technique was found which moves one real Poincaré exponent to any other real value while leaving all other Poincaré exponents undisturbed. In addition, for the scalar case, another technique has been demonstrated which allows for changing the sum of two or more Poincaré exponents. This technique is quite useful when adding damping to a pair of complex-conjugate exponents. However, unlike the single exponent scalar case, this technique does not assure stability, and requires a Floquet analysis of the controlled system to determine the actual system exponents. Both of these techniques require making the control proportional to the adjoint eigenvector associated with the Poincaré exponent to be moved. Using vector control, an algorithm for moving n Poincaré exponents independently has been presented, where n is less than or equal to the rank of the modal controllability matrix. Again, the remaining Poincaré exponents are left unchanged. The vector algorithm allows the user to explicitly specify the desired, stable, closed-loop system matrix.

In summary, several very practical techniques for the control of time-periodic linear systems have been demonstrated. Methods have been developed which allow the solution to a broad range of interesting problems. Much work remains to be done, however, both in extending the theory to more general cases, and in gaining experience with actual applications of the vector control techniques.

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